BUTP-94/9

Virtual Photons in Chiral Perturbation Theory#

Res Urech
Institute for Theoretical Physics, University of Berne
Sidlerstr. 5, CH-3012 Berne, Switzerland

e-mail: urech@itp.unibe.ch

Abstract

In the framework of chiral perturbation theory virtual photons are included. We calculate the divergences of the generating functional to one loop and determine the structure of the local action that incorporates the counterterms which cancel the divergences. As an application we discuss the corrections to Dashen's theorem at order $e^2 m_q$.

May 1994

#Work supported in part by Schweizerischer Nationalfonds.

1 Introduction

In the chiral limit where the light up, down and strange quark masses go to zero, the QCD lagrangian has a $SU(3)_R \times SU(3)_L$ chiral symmetry that is spontaneously broken by the groundstate of the theory . There are eight Goldstone bosons: The π's, K's and η. Their interactions are described by an effective low energy theory, called chiral perturbation theory (CHPT). At low momenta the chiral lagrangian can be expanded in derivatives of the Goldstone fields and in the masses of the three light quarks.

CHPT is a nonrenormalizable theory: Loops produce ultraviolet divergences, which can be absorbed by introducing counterterms. The finite parts of the coupling constants of this counterterm lagrangian are not determined by the lagrangian that generated the loops. The complete information about them is hidden in the QCD lagrangian, but there is no known way to extract this information from first principle alone. The couplings have to be evaluated from experiments. As shown by Weinberg [1], the loops are suppressed: Every loop and the associated counterterm correspond to successively higher powers of momenta or quark masses. At low energies, the contributions from higher loops are small.

CHPT is an effective theory: It contains all terms allowed by the symmetry of the QCD lagrangian in the chiral limit. The local action is generated by the effective lagrangian, which is in general not determined by the counterterms alone at higher orders in the expansion of momenta and quark masses. Additional terms with finite couplings have to be included.

The generating functional to one loop and the corresponding local action in the strong sector has been determined by Gasser and Leutwyler [2]. An extension to the $\Delta S = 1$ nonleptonic weak interactions to one loop has been given by Kambor, Missimer and Wyler [3]. Here we consider virtual photons and include them in the mesonic sector. We calculate the divergences of the one-loop functional in presence of external currents and determine the structure of the local action at this order.

The article is organized as follows. In the second chapter virtual photons are included in CHPT with three flavours. We evaluate the divergences of the generating functional to one loop and determine the structure of the local action at order p^4. In the third chapter we discuss the corrections to Dashen's theorem at order $e^2 m_q$ and compare our estimates with the results in the literature. In the fourth chapter we summarize our results and in Appendix A the matrix relations are given that are used to simplify the effective lagrangian at order p^4. Finally, in Appendix B we list the renormalized masses of the pseudoscalar mesons at order $e^2 m_q$.

2 Generating Functional

We suppose that the reader is familiar with CHPT, otherwise we refer her or him to comprehensive reviews [4].

At leading order, the effective lagrangian of the strong and electromagnetic interactions which respects the chiral symmetry $SU(3)_R \times SU(3)_L$, P and C invariance is in the mesonic sector [5]

$$\begin{aligned} \mathcal{L}_2^{(Q)} = & - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2} (\partial_\mu A^\mu)^2 \\ & + \frac{1}{4} \mathrm{F}_\mathrm{o}^2 < d^\mu U^+ d_\mu U + \chi U^+ + \chi^+ U > + \, \mathrm{C} < QUQU^+ >, \end{aligned} \tag{1}$$

where $F_{\mu\nu}$ is the field strength tensor of the photon field A_μ, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. λ is the gauge fixing parameter and will from now on be kept at $\lambda = 1$. U is an unitary 3×3-matrix and incorporates the fields of the eight pseudoscalar mesons,

$$\begin{aligned} & UU^+ = \mathbf{1}, && \det U = 1, \\ & U = exp\,(i\Phi/\mathrm{F}_\mathrm{o}), && \Phi = \sum_{a=1}^{8} \lambda_a \varphi_a \ , \end{aligned} \tag{2}$$

where the λ_a's are the Gell-Mann matrices, and

$$\Phi = \sqrt{2} \begin{pmatrix} \frac{\pi^\mathrm{o}}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^\mathrm{o}}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^\mathrm{o} \\ K^- & \bar{K}^\mathrm{o} & -\frac{2\eta_8}{\sqrt{6}} \end{pmatrix}. \tag{3}$$

$d_\mu U$ is a covariant derivative, incorporating the couplings to the photon field A_μ, the external vector and axial vector currents v_μ and a_μ, respectively,

$$d_\mu U = \partial_\mu U - i(v_\mu + QA_\mu + a_\mu)U + iU(v_\mu + QA_\mu - a_\mu), \tag{4}$$

where Q is the charge matrix of the three light quarks,

$$Q = \frac{e}{3} \begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix} = \frac{e}{2} \left(\lambda_3 + \frac{1}{\sqrt{3}} \lambda_8 \right). \tag{5}$$

We do not consider singlet vector and axial vector currents and we put therefore tr $v_\mu =$ tr $a_\mu = 0$. χ denotes the coupling of the mesons to the scalar and pseudoscalar currents s and p, respectively,

$$\chi = 2B_\mathrm{o}(s + ip), \tag{6}$$

where s incorporates the mass matrix of the quarks,

$$s = \mathcal{M} + \cdots = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} + \cdots \tag{7}$$

F_o is the pion decay constant in the chiral limit, $\mathrm{F}_\pi = \mathrm{F}_\mathrm{o}\,[1 + O(m_q)]$. B_o is related to the quark condensate $< \mathrm{o}|\bar{u}u|\mathrm{o} >= -\mathrm{F}_\mathrm{o}^2 B_\mathrm{o}\,[1 + O(m_q)]$, and C determines the purely electromagnetic part of the masses of the charged pions and kaons in the chiral limit,

$$M_{\pi^\pm}^2 = M_{K^\pm}^2 = 2e^2 \frac{\mathrm{C}}{\mathrm{F}_\mathrm{o}^2} + O(m_q). \tag{8}$$

To ensure the chiral $SU(3)_R \times SU(3)_L$ symmetry of the lagrangian $\mathcal{L}_2^{(Q)}$ we introduce local spurions Q_R, Q_L instead of the charge matrix Q. The lagrangian is modified,

$$\begin{aligned}\mathcal{L}_2^{(Q)} = & - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2 \\ & + \frac{1}{4} \mathrm{F}_\mathrm{o}^2 < d^\mu U^+ d_\mu U + \chi U^+ + \chi^+ U > + \mathrm{C} < Q_R U Q_L U^+ >, \end{aligned} \tag{9}$$

and $d_\mu U$ is changed to

$$d_\mu U = \partial_\mu U - i(v_\mu + Q_R A_\mu + a_\mu) U + iU(v_\mu + Q_L A_\mu - a_\mu). \tag{10}$$

The rules of chiral transformation are [2, 5]

$$\begin{aligned} U & \rightarrow g_R U g_L^+, \\ Q_I & \rightarrow g_I Q_I g_I^+, \qquad\qquad I = R, L \\ v_\mu + Q_R A_\mu + a_\mu & \rightarrow g_R (v_\mu + Q_R A_\mu + a_\mu) g_R^+ + i g_R \partial_\mu g_R^+, \\ v_\mu + Q_L A_\mu - a_\mu & \rightarrow g_L (v_\mu + Q_L A_\mu - a_\mu) g_L^+ + i g_L \partial_\mu g_L^+, \\ s + ip & \rightarrow g_R (s + ip) g_L^+, \\ g_{R,L} & \in SU(3)_{R,L} \,. \end{aligned} \tag{11}$$

Since the currents v_μ and a_μ count as $O(p)$, where p means the momentum of the external fields, the term QA_μ has the same dimension. We put

$$dim(Q_{R,L}) = O(p), \qquad\qquad dim(A_\mu) = O(1), \tag{12}$$

the lagrangian $\mathcal{L}_2^{(Q)}$ is therefore of order p^2. The convention (12) has the advantage that electromagnetic interactions do not turn upside down the usual chiral counting. The procedure to obtain the generating functional at order $O(p^4)$ is very similar to conventional CHPT. The one-loop graphs generated by $\mathcal{L}_2^{(Q)}$ are of order $O(p^4)$. They contain divergences which are absorbed by adding tree graphs, evaluated with the lagrangian $\mathcal{L}_4^{(Q)}$ of order $O(p^4)$, see below. The generating functional becomes, up to and including terms of order $O(p^4)$,

$$e^{i\mathcal{Z}(v_\mu, a_\mu, s, p)} = N \int [dU]\, [dA_\mu]\, e^{i \int d^4x \left\{ \mathcal{L}_2^{(Q)} + \mathcal{L}_4^{(Q)} \right\}}, \tag{13}$$

where the integration over the fields is carried out in the one-loop approximation. [Here and below we disregard contributions from the anomaly altogether.] To evaluate the divergent part of the one-loop functional, we transform the lagrangian $\mathcal{L}_2^{(Q)}$ (9) to Euclidean spacetime, $\mathcal{L}_2^{(Q)} \rightarrow \mathcal{L}_E^{(Q)}$, and expand the fields U and A_μ around the classical solutions $(\bar{U}, \bar{A}_\mu)$ of the equations of motion,

$$\begin{aligned} U & = u e^{i\xi/\mathrm{F}_\mathrm{o}} u = u \left(\mathbf{1} + i \frac{\xi}{\mathrm{F}_\mathrm{o}} - \frac{1}{2} \frac{\xi^2}{\mathrm{F}_\mathrm{o}^2} + \cdots \right) u \\ & = \bar{U} + \frac{i}{\mathrm{F}_\mathrm{o}} u \xi u - \frac{1}{2\mathrm{F}_\mathrm{o}^2} u \xi^2 u + \cdots \\ A_\mu & = \bar{A}_\mu + \epsilon_\mu, \end{aligned} \tag{14}$$

where we put $\bar{U} = u^2$, and ξ is a traceless hermitean matrix. We insert the expansion (14) in the action $\mathcal{S}_E$,

$$
\begin{aligned}
\mathcal{S}_E \quad = \quad & \int d^4x_E \, \bar{\mathcal{L}}_E^{(Q)} \\
& + \int d^4x_E \left\{ \frac{1}{4} < D_\mu \xi D_\mu \xi - [\Delta_\mu, \xi][\Delta_\mu, \xi] + \sigma \xi^2 > \right. \\
& \qquad - \frac{1}{8} \frac{\mathrm{C}}{\mathrm{F}_\mathrm{o}^2} < [H_R + H_L, \xi][H_R - H_L, \xi] > \\
& \qquad + \frac{1}{2} \mathrm{F}_0 < (\xi [H_R, \Delta_\mu] - H_L D_\mu \xi) > \epsilon_\mu \\
& \qquad \left. + \frac{1}{2} \epsilon_\mu \left(-\partial_\nu \partial_\nu + \frac{1}{2} \mathrm{F}_\mathrm{o}^2 < H_L^2 > \right) \epsilon_\mu \right\} + \cdots, \qquad (15)
\end{aligned}
$$

where the ellipsis denotes higher order terms in the fluctuations ξ and ϵ_μ. We used the expressions (see also ref.[2])

$$
\begin{aligned}
D_\mu \xi &= \partial_\mu \xi + [\Gamma_\mu, \xi], \\
\Gamma_\mu &= \frac{1}{2}[u^+, \partial_\mu u] - \frac{1}{2} i u^+ G_\mu^R u - \frac{1}{2} i u G_\mu^L u^+, \\
\Delta_\mu &= \frac{1}{2} u^+ d_\mu \bar{U} u^+ = -\frac{1}{2} u d_\mu \bar{U}^+ u, \\
G_\mu^R &= v_\mu + Q_R \bar{A}_\mu + a_\mu, \\
G_\mu^L &= v_\mu + Q_L \bar{A}_\mu - a_\mu, \\
H_R &= u^+ \{Q, \bar{U}\} u^+ = u^+ Q_R u + u Q_L u^+, \\
H_L &= u^+ [Q, \bar{U}] u^+ = u^+ Q_R u - u Q_L u^+, \\
\sigma &= \frac{1}{2}(u^+ \chi u^+ + u \chi^+ u). \qquad (16)
\end{aligned}
$$

Γ_μ and Δ_μ are antihermitean matrices, whereas $H_{R,L}$ and σ are hermitean ones and $\int d^4x_E \, \bar{\mathcal{L}}_E^{(Q)}$ represents the classical action of $\mathcal{L}_E^{(Q)}$. The normal brackets $[\cdots]$ denote the commutator, the curly brackets $\{\cdots\}$ stand for the anti-commutator. We use the parametrization $\xi = \sum_a \xi^a \lambda^a$, where the λ^a's are the Gell-Mann matrices. We define a new covariant derivative $\Sigma_\mu = D_\mu + X_\mu$ with

$$
\begin{aligned}
\Sigma_\mu \xi^a &= D_\mu \xi^a + X_\mu^{a\rho} \epsilon^\rho \, , \\
\Sigma_\mu \epsilon^\rho &= \partial_\mu \epsilon^\rho + X_\mu^{\rho a} \xi^a \, , \qquad (17)
\end{aligned}
$$

where $D_\mu \xi^a$ is understood as

$$
D_\mu \xi^a = \partial_\mu \xi^a + \Gamma_\mu^{ab} \xi^b \qquad (18)
$$

and

$$
\Gamma_\mu^{ab} \quad = \quad -\frac{1}{2} < [\lambda^a, \lambda^b] \Gamma_\mu >,
$$

$$X_\mu^{a\rho} = -X_\mu^{\rho a} = -\frac{1}{4}\mathrm{F}_0 < H_L\lambda^a > \delta_\mu^\rho. \tag{19}$$

The action becomes at the one-loop level

$$\begin{aligned}\mathcal{S}_E|_{one\ loop} &= \frac{1}{2}\int d^4x_E \left\{\xi^a\left(-\Sigma_\mu\Sigma_\mu\delta^{ab} + \sigma^{ab}\right)\xi^b + \xi^a\gamma_\mu^a\epsilon_\mu \right.\\ &\quad \left. +\epsilon_\mu\left(-\Sigma_\nu\Sigma_\nu + \rho\right)\epsilon_\mu\right\},\end{aligned} \tag{20}$$

where now

$$\begin{aligned}\sigma^{ab} &= -\frac{1}{2} < [\Delta_\mu,\lambda^a][\Delta_\mu,\lambda^b] > + \frac{1}{4} < \sigma\{\lambda^a,\lambda^b\} > \\ &\quad -\frac{1}{4}\frac{\mathrm{C}}{\mathrm{F}_\mathrm{o}^2} < [H_R + H_L,\lambda^a][H_R - H_L,\lambda^b] > -\frac{1}{4}\mathrm{F}_\mathrm{o}^2 < H_L\lambda^a >< H_L\lambda^b >,\\ \gamma_\mu^a &= \mathrm{F}_\mathrm{o} < \left([H_R,\Delta_\mu] + \frac{1}{2}D_\mu H_L\right)\lambda^a >,\\ \rho &= \frac{3}{8}\mathrm{F}_\mathrm{o}^2 < H_L^2 > .\end{aligned} \tag{21}$$

We collect the fluctuations of the mesons and of the photon field and define a new flavour space η^A, where A runs from 1 to 12, $\eta = (\xi^1,\ldots,\xi^8,\epsilon^0,\ldots,\epsilon^3)$. The action above can now be written as a quadratic form,

$$\mathcal{S}_E|_{one\ loop} = \frac{1}{2}\int d^4x_E\ \eta^A\left(-\Sigma_\mu\Sigma_\mu\delta^{AB} + \Lambda^{AB}\right)\eta^B, \tag{22}$$

where Σ_μ and Λ^{AB} are 12×12-matrices,

$$\begin{aligned}\Sigma_\mu &= \partial_\mu\mathbf{1} + \begin{pmatrix}\Gamma_\mu^{ab} & X_\mu^{a\rho}\\ X_\mu^{\sigma b} & 0\end{pmatrix} = \partial_\mu\mathbf{1} + Y_\mu,\\ \Lambda &= \begin{pmatrix}\sigma^{ab} & \frac{1}{2}\gamma^{a\rho}\\ \frac{1}{2}\gamma^{\sigma b} & \rho\delta^{\sigma\rho}\end{pmatrix}.\end{aligned} \tag{23}$$

This leads to a gaussian integral for the generating functional,

$$\begin{aligned}e^{-\mathcal{Z}_E|_{one\ loop}} &= N\int d\mu[\eta]e^{-\frac{1}{2}(\eta,\mathcal{D}\eta)}\\ &= N'(\det\mathcal{D})^{-\frac{1}{2}},\end{aligned} \tag{24}$$

where we defined the differential operator $\mathcal{D}^{AB} = -\Sigma_\mu\Sigma_\mu\delta^{AB} + \Lambda^{AB}$ and

$$(f,g) = \sum_A\int d^4x_E f^A g^A. \tag{25}$$

Omitting the constant contribution we arrive at

$$\mathcal{Z}_E|_{one\ loop} = \frac{1}{2}\ln(\det\mathcal{D}). \tag{26}$$

	P	C
U	U^+	U^T
$d_\mu U$	$d^\mu U^+$	$(d_\mu U)^T$
G_μ^R	$G^{L\mu}$	${G_\mu^L}^T$
G_μ^L	$G^{R\mu}$	${G_\mu^R}^T$
χ	χ^+	χ^T
Q_R	Q_L	Q_L^T
$c_\mu^R Q_R$	$c^{L\mu} Q_L$	$(c_\mu^L Q_L)^T$
Q_L	Q_R	Q_R^T
$c_\mu^L Q_L$	$c^{R\mu} Q_R$	$(c_\mu^R Q_R)^T$

Table 1: *P and C transformation properties for the fields, the charges and for their derivatives. The defintion of $c_\mu^R Q_R$ and $c_\mu^L Q_L$ is given in the text. Spacetime arguments are suppressed.*

To renormalize the determinant we use dimensional regularization. In Minkowski spacetime, the one-loop functional in $d = 4$ dimensions is given by [6]

$$\mathcal{Z}_{one\ loop} = -\frac{1}{16\pi^2}\frac{1}{d-4}\int d^4x \ \mathrm{Tr}\left(\frac{1}{12}Y_{\mu\nu}Y^{\mu\nu} + \frac{1}{2}\Lambda^2\right) + \text{ finite parts,} \tag{27}$$

where Tr means the trace in the flavour space η^A and $Y_{\mu\nu}$ denotes the field strength tensor of Y_μ,

$$Y_{\mu\nu} = \partial_\mu Y_\nu - \partial_\nu Y_\mu + [Y_\mu, Y_\nu]. \tag{28}$$

In table 1 we list P and C transformation properties for the fields and quantities used in the one-loop functional. $c_\mu^R Q_R$ and $c_\mu^L Q_L$ mean covariant derivatives of Q_R and Q_L, respectively,

$$c_\mu^I Q_I = \partial_\mu Q_I - i[G_\mu^I, Q_I], \qquad\qquad I = R, L\ , \tag{29}$$

that transform under $SU(3)_R \times SU(3)_L$ in the same way as Q_R and Q_L,

$$\begin{aligned} c_\mu^I Q_I &\rightarrow g_I\ c_\mu^I Q_I\ g_I^+, \qquad\qquad I = R, L \\ g_{R,L} &\in SU(3)_{R,L}\,. \end{aligned} \tag{30}$$

The divergences in $\mathcal{Z}_{one\ loop}$ can be absorbed by adding the effective lagrangian $\mathcal{L}_4^{(Q)}$ that contains all terms of order $O(p^4)$ allowed by the chiral symmetry, P and C invariance. Since in the usual

physical picture the charge matrix is a constant and has no chirality, we put again $Q_R = Q_L = Q$ and $\partial_\mu Q = 0$. We keep the notation $c_\mu^{R,L} Q = -i[G_\mu^{R,L}, Q]$ in order to remember the correct chiral transformation of $c_\mu^I Q_I$. By using the matrix relations that we list in Appendix A, the lagrangian $\mathcal{L}_4^{(Q)}$ can be simplified to

$$
\begin{aligned}
\mathcal{L}_4^{(Q)} = \; & L_1 < d^\mu \bar{U}^+ d_\mu \bar{U} >^2 \; + \; L_2 < d^\mu \bar{U}^+ d^\nu \bar{U} >< d_\mu \bar{U}^+ d_\nu \bar{U} > \\
& + \; L_3 < d^\mu \bar{U}^+ d_\mu \bar{U} d^\nu \bar{U}^+ d_\nu \bar{U} > \; + \; L_4 < d^\mu \bar{U}^+ d_\mu \bar{U} >< \chi^+ \bar{U} + \chi \bar{U}^+ > \\
& + \; L_5 < d^\mu \bar{U}^+ d_\mu \bar{U} (\chi^+ \bar{U} + \bar{U}^+ \chi) > \; + \; L_6 < \chi^+ \bar{U} + \chi \bar{U}^+ >^2 \; + \; L_7 < \chi^+ \bar{U} - \chi \bar{U}^+ >^2 \\
& + \; L_8 < \chi^+ \bar{U} \chi^+ \bar{U} + \chi \bar{U}^+ \chi \bar{U}^+ > \; - \; i L_9 < d^\mu \bar{U} d^\nu \bar{U}^+ G^R_{\mu\nu} + d^\mu \bar{U}^+ d^\nu \bar{U} G^L_{\mu\nu} > \\
& + \; L_{10} < G^{R\,\mu\nu} \bar{U} G^L_{\mu\nu} \bar{U}^+ > \; + \; H_1 < G^{R\,\mu\nu} G^R_{\mu\nu} + G^{L\mu\nu} G^L_{\mu\nu} > \; + \; H_2 < \chi^+ \chi > \\
& + \; K_1 \mathrm{F}_\mathrm{o}^2 < d^\mu \bar{U}^+ d_\mu \bar{U} >< Q^2 > \; + \; K_2 \mathrm{F}_\mathrm{o}^2 < d^\mu \bar{U}^+ d_\mu \bar{U} >< Q \bar{U} Q \bar{U}^+ > \\
& + \; K_3 \mathrm{F}_\mathrm{o}^2 \left(< d^\mu \bar{U}^+ Q \bar{U} >< d_\mu \bar{U}^+ Q \bar{U} > + < d^\mu \bar{U} Q \bar{U}^+ >< d_\mu \bar{U} Q \bar{U}^+ > \right) \\
& + \; K_4 \mathrm{F}_\mathrm{o}^2 < d^\mu \bar{U}^+ Q \bar{U} >< d_\mu \bar{U} Q \bar{U}^+ > + \; K_5 \mathrm{F}_\mathrm{o}^2 < \left(d^\mu \bar{U}^+ d_\mu \bar{U} + d^\mu \bar{U} d_\mu \bar{U}^+ \right) Q^2 > \\
& + \; K_6 \mathrm{F}_\mathrm{o}^2 < d^\mu \bar{U}^+ d_\mu \bar{U} Q \bar{U}^+ Q \bar{U} + d^\mu \bar{U} d_\mu \bar{U}^+ Q \bar{U} Q \bar{U}^+ > \\
& + \; K_7 \mathrm{F}_\mathrm{o}^2 < \chi^+ \bar{U} + \bar{U}^+ \chi >< Q^2 > \; + \; K_8 \mathrm{F}_\mathrm{o}^2 < \chi^+ \bar{U} + \bar{U}^+ \chi >< Q \bar{U} Q \bar{U}^+ > \\
& + \; K_9 \mathrm{F}_\mathrm{o}^2 < (\chi^+ \bar{U} + \bar{U}^+ \chi + \chi \bar{U}^+ + \bar{U} \chi^+) Q^2 > \\
& + \; K_{10} \mathrm{F}_\mathrm{o}^2 < (\chi^+ \bar{U} + \bar{U}^+ \chi) Q \bar{U}^+ Q \bar{U} + (\chi \bar{U}^+ + \bar{U} \chi^+) Q \bar{U} Q \bar{U}^+ > \\
& + \; K_{11} \mathrm{F}_\mathrm{o}^2 < (\chi^+ \bar{U} - \bar{U}^+ \chi) Q \bar{U}^+ Q \bar{U} + (\chi \bar{U}^+ - \bar{U} \chi^+) Q \bar{U} Q \bar{U}^+ > \\
& + \; K_{12} \mathrm{F}_\mathrm{o}^2 < d^\mu \bar{U}^+ \left[c_\mu^R Q, Q \right] \bar{U} + d^\mu \bar{U} \left[c_\mu^L Q, Q \right] \bar{U}^+ > \\
& + \; K_{13} \mathrm{F}_\mathrm{o}^2 < c^{R\mu} Q \bar{U} c_\mu^L Q \bar{U}^+ > \; + \; K_{14} \mathrm{F}_\mathrm{o}^2 < c^{R\mu} Q \; c_\mu^R Q + c^{L\mu} Q \; c_\mu^L Q > \\
& + \; K_{15} \mathrm{F}_\mathrm{o}^4 < Q \bar{U} Q \bar{U}^+ >^2 + \; K_{16} \mathrm{F}_\mathrm{o}^4 < Q \bar{U} Q \bar{U}^+ >< Q^2 > \; + \; K_{17} \mathrm{F}_\mathrm{o}^4 < Q^2 >^2 . \qquad (31)
\end{aligned}
$$

where $G^R_{\mu\nu}$ and $G^L_{\mu\nu}$ are the field strength tensors of G_μ^R, G_μ^L, respectively,

$$
G^I_{\mu\nu} \;\; = \;\; \partial_\mu G^I_\nu - \partial_\nu G^I_\mu - i \left[G^I_\mu, G^I_\nu \right], \qquad\qquad I = R, L. \qquad (32)
$$

If we put $Q = 0$, the terms with the couplings K_i vanish and we are left with the contribution to $\mathcal{L}_4^{(Q)}$ from the purely strong sector, calculated by Gasser and Leutwyler [2]. In (31) we have only considered contributions made out of the building blocks in table 1. In particular, terms proportional to polynomials in $\bar{A}_\mu$ are omitted. The coupling constants L_i, H_i and K_i are defined as

$$
\begin{aligned}
L_i &= \Gamma_i \lambda + L_i^r(\mu), \\
H_i &= \Delta_i \lambda + H_i^r(\mu), \\
K_i &= \Sigma_i \lambda + K_i^r(\mu). \qquad (33)
\end{aligned}
$$

λ contains a pole in $d = 4$ dimensions,

$$\lambda = \frac{\mu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} - \frac{1}{2}\left[\ln 4\pi + \Gamma'(1) + 1\right]\right\}. \tag{34}$$

The coefficients Γ_i, Δ_i and the finite parts $L_i^r(\mu)$ are listed in ref.[2], whereas H_1^r and H_2^r are of no physical significance. The coefficients Σ_i are

$$\begin{array}{lll}
\Sigma_1 = \frac{3}{4} & \Sigma_2 = Z & \Sigma_3 = -\frac{3}{4} \\
\Sigma_4 = 2Z & \Sigma_5 = -\frac{9}{4} & \Sigma_6 = \frac{3}{2}Z \\
\Sigma_7 = 0 & \Sigma_8 = Z & \Sigma_9 = -\frac{1}{4} \\
\Sigma_{10} = \frac{1}{4} + \frac{3}{2}Z & \Sigma_{11} = \frac{1}{8} & \Sigma_{12} = \frac{1}{4} \\
\Sigma_{13} = 0 & \Sigma_{14} = 0 & \Sigma_{15} = \frac{3}{2} + 3Z + 20Z^2 \\
\Sigma_{16} = -3 - \frac{3}{2}Z - 4Z^2 & \Sigma_{17} = \frac{3}{2} - \frac{3}{2}Z + 2Z^2 &
\end{array} \tag{35}$$

with

$$Z = \frac{\mathrm{C}}{\mathrm{F_o^4}}. \tag{36}$$

We choose the renormalized coefficients $K_i^r(\mu)$ in such a way that the K_i themselves do not depend on the scale μ,

$$\mu\frac{d}{d\mu}K_i = \Sigma_i\frac{\mu^{d-4}}{16\pi^2} + \mu\frac{d}{d\mu}K_i^r(\mu) + O(d-4) = 0. \tag{37}$$

3 Corrections to Dashen's Theorem

This chapter is separated in four sections. In the first section we calculate the formal expression for the corrections to Dashen's theorem at order $e^2 m_q$, i.e. the squared mass difference $(M_{K^\pm}^2 - M_{K^\circ}^2) - (M_{\pi^\pm}^2 - M_{\pi^\circ}^2)$. In the second section a numerical estimate is given with an upper limit on the unknown terms. In the third section we give an independent approach on the basis of ref.[7]. Finally, we compare in the fourth section the obtained values with the results in the literature.

3.1 Masses at Order $e^2 m_q$

In the chiral limit, the masses of the pions and the kaons are of purely electromagnetic nature. A relation between the squared masses at order e^2 is given by Dashen's theorem [8],

$$\left(M_{\pi^\pm}^2 - M_{\pi^\circ}^2\right)_{e.m.} = \left(M_{K^\pm}^2 - M_{K^\circ}^2\right)_{e.m.} \tag{38}$$

with

$$\begin{aligned}
M_{\pi^\pm}^2 &= M_{K^\pm}^2 = 2e^2\frac{\mathrm{C}}{\mathrm{F_o^2}}, \\
M_{\pi^\circ}^2 &= M_{K^\circ}^2 = 0 \qquad\qquad \text{for} \quad m_q = 0.
\end{aligned} \tag{39}$$

At lowest order in the quark mass expansion the squared masses of the pseudoscalar mesons are denoted by $\hat{M}_P^2$. We neglect the mass difference $m_d - m_u$ and replace m_u, m_d by $\hat{m} = (m_u + m_d)/2$. If we expand the lagrangian $\mathcal{L}_2^{(Q)}$ in the parametrization (2) and use the representation (3) of the physical fields, the squared masses are

$$
\begin{aligned}
\hat{M}^2_{\pi^\pm} &= 2e^2\frac{\mathrm{C}}{\mathrm{F}_\mathrm{o}^2} + 2\hat{m}B_\mathrm{o}\,, \\
\hat{M}^2_{\pi^\mathrm{o}} &= \hat{M}^2_\pi = 2\hat{m}B_\mathrm{o}\,, \\
\hat{M}^2_{K^\pm} &= 2e^2\frac{\mathrm{C}}{\mathrm{F}_\mathrm{o}^2} + (\hat{m}+m_s)B_\mathrm{o}\,, \\
\hat{M}^2_{K^\mathrm{o}} &= \hat{M}^2_K = (\hat{m}+m_s)B_\mathrm{o}\,, \\
\hat{M}^2_\eta &= \frac{2}{3}(\hat{m}+2m_s)B_\mathrm{o}\,.
\end{aligned}
\tag{40}
$$

This decomposition depends on the convention adopted for the quark self-energies. The effect of this ambiguity is small and neglected in this article. The correction to Dashen's theorem at order $e^2 m_q$ is denoted by

$$
\Delta M_K^2 - \Delta M_\pi^2 = (M^2_{K^\pm} - M^2_{K^\mathrm{o}}) - (M^2_{\pi^\pm} - M^2_{\pi^\mathrm{o}}) \qquad \text{for} \quad m_u = m_d = \hat{m}. \tag{41}
$$

We calculate the Fourier transform of the two-point function of the mesons to one loop, keeping the fields associated with the Gell-Mann matrices λ_a instead of the physical ones. For the charged fields it has a cut at $p^2 = M_a^2$,

$$
\begin{aligned}
&i\int d^4x\, e^{ipx} < \mathrm{o}|T\varphi_a(x)\varphi_a(\mathrm{o})e^{i\int d^4y\left\{\mathcal{L}^{(Q)}_{2\,int}(y)+\mathcal{L}^{(Q)}_4(y)\right\}}|\mathrm{o}>|_{to\ one\ loop} \\
&= \frac{\tilde{Z}_a(M_a^2)}{M_a^2(1-p^2/M_a^2)^{1+e^2 f_a(p^2)}} + \cdots \qquad a = 1,2,4,5
\end{aligned}
\tag{42}
$$

with

$$
f_a(p^2) = \frac{1}{8\pi^2}\left(1 + \frac{M_a^2}{p^2}\right), \tag{43}
$$

whereas for the neutral fields it contains a pole,

$$
\begin{aligned}
&i\int d^4x\, e^{ipx} < \mathrm{o}|T\varphi_a(x)\varphi_a(\mathrm{o})e^{i\int d^4y\left\{\mathcal{L}^{(Q)}_{2\,int}(y)+\mathcal{L}^{(Q)}_4(y)\right\}}|\mathrm{o}>|_{to\ one\ loop} \\
&= \frac{Z_a(M_a^2)}{M_a^2 - p^2} + \cdots \qquad a = 3,6,7,8.
\end{aligned}
\tag{44}
$$

$\mathcal{L}^{(Q)}_{2\,int}$ represents the interaction part of the lagrangian $\mathcal{L}_2^{(Q)}$, $Z_a(M_a^2)$ and $\tilde{Z}_a(M_a^2)$ are due to the field renormalization and the ellipses denote terms which are not relevant for the mass-shifts at order $e^2 m_q$. The contributions to the two-point function are shown in fig. 1 : (b) and (c) are one-photon loops. The first contributes to the two-point function of the charged fields only, the latter vanishes in dimensional regularization. (d) is the tadpole with a four-meson coupling and (e) represents the

contributions from the lagrangian $\mathcal{L}_4^{(Q)}$. The results for the different mesons are explicitly given in Appendix B. Here we consider only the difference $\Delta M_K^2 - \Delta M_\pi^2$,

$$\begin{aligned}
\Delta M_K^2 - \Delta M_\pi^2 &= -e^2 \frac{1}{16\pi^2}\left[3M_K^2 \ln\frac{M_K^2}{\mu^2} - 3M_\pi^2 \ln\frac{M_\pi^2}{\mu^2} - 4(M_K^2 - M_\pi^2)\right] \\
&\quad -e^2 \frac{1}{8\pi^2}\frac{\mathrm{C}}{\mathrm{F}_\mathrm{o}^4}\left[M_K^2 \ln\frac{M_K^2}{\mu^2} - M_\pi^2(2\ln\frac{M_\pi^2}{\mu^2} + 1)\right] \\
&\quad -16e^2\frac{\mathrm{C}}{\mathrm{F}_\mathrm{o}^4} L_5^r(\mu)(M_K^2 - M_\pi^2) + R_\pi(\mu)M_\pi^2 + R_K(\mu)M_K^2 \\
&\quad +O(e^2 m_q^2),
\end{aligned} \tag{45}$$

where $L_5^r(\mu)$ and $R_{\pi,K}$ are the contributions from $\mathcal{L}_4^{(Q)}$,

$$\begin{aligned}
R_\pi &= \frac{2}{3}e^2\left(6K_3^r - 3K_4^r + 2K_9^r - 10K_{10}^r - 12K_{11}^r\right), \\
R_K &= -\frac{4}{3}e^2\left(K_5^r + K_6^r - 6K_{10}^r - 6K_{11}^r\right).
\end{aligned} \tag{46}$$

At the end we need L_5^r and seven coupling constants K_i^r from $\mathcal{L}_4^{(Q)}$ to determine the correction to Dashen's theorem at order $e^2 m_q$.

3.2 Numerical Results

We put F_o equal to the physical pion decay constant, $\mathrm{F}_\pi = 93.3\,\mathrm{MeV}$ and the masses of the mesons to $\hat{M}_\pi = 135\,\mathrm{MeV}$, $\hat{M}_K = 495\,\mathrm{MeV}$. C can be expressed as an integral over the difference of the vector and axial vector spectral functions as established by Das, Guralnik, Mathur, Low and Young [9]. If we consider the low-lying vector and axial-vector mesons ρ and A_1, respectively, and use the Weinberg sum rules [10] to eliminate the parameters of the A_1, we arrive at [11]

$$\mathrm{C} = \frac{3}{32\pi^2} M_\rho^2 \mathrm{F}_\rho^2 \ln\left(\frac{\mathrm{F}_\rho^2}{\mathrm{F}_\rho^2 - \mathrm{F}_\pi^2}\right), \tag{47}$$

where M_ρ is the mass of the ρ meson, $M_\rho = 770\,\mathrm{MeV}$, and F_ρ denotes the ρ decay constant, $\mathrm{F}_\rho = 154\,\mathrm{MeV}$ [5]. Therefore

$$\mathrm{C} = 61.1 \times 10^{-6}\,(\mathrm{GeV})^4. \tag{48}$$

At tree level, the difference of the squared pion masses is

$$\begin{aligned}
\Delta\hat{M}_\pi^2 &= (\hat{M}_{\pi^\pm} + \hat{M}_{\pi^\circ})(\hat{M}_{\pi^\pm} - \hat{M}_{\pi^\circ}) \\
&= 2M_\pi \times 4.8\,\mathrm{MeV},
\end{aligned} \tag{49}$$

which is very close to the experimental value, $(M_{\pi^\pm} - M_{\pi^\circ})_{exp.} = 4.6\,\mathrm{MeV}$ [12]. The spectral functions can also be extracted from data of the τ-decay [13]. However, this method contains uncertainties because the spectral functions can only be evaluated for momenta $p^2 \leq 2(\mathrm{GeV})^2$ and

therefore we keep our approach above. $L_5^r(\mu)$ is taken from table 1 of ref.[5]. Its values at the scale points $\mu = (0.5\,\mathrm{GeV};\ M_\rho;\ 1\,\mathrm{GeV})$ are

$$L_5^r(\mu) = (2.4;\ 1.4;\ 0.8) \pm 0.5 \times 10^{-3}. \tag{50}$$

These values lead to the following result for the squared mass differences, using again the three scales $\mu = (0.5\,\mathrm{GeV};\ M_\rho;\ 1\,\mathrm{GeV})$,

$$\Delta M_K^2 - \Delta M_\pi^2 = (-0.26;\ 0.52;\ 0.99) \pm 0.13 \times 10^{-3} (\mathrm{GeV})^2\ + R_\pi M_\pi^2 + R_K M_K^2. \tag{51}$$

There is a strong dependence on the scale μ for all the terms. Of course, this scale-dependence cancels in the full expression.

Next we calculate the ratio $(\Delta M_K^2/\Delta M_\pi^2)$. For this purpose we put $\Delta M_\pi^2 = (M_{\pi^\pm}^2 - M_{\pi^o}^2)_{exp.}$ and arrive at

$$\frac{\Delta M_K^2}{\Delta M_\pi^2} = (0.79;\ 1.42;\ 1.80) \pm 0.11 + \frac{R_\pi M_\pi^2 + R_K M_K^2}{\Delta M_\pi^2}. \tag{52}$$

In ΔM_π^2 we can estimate the coupling constant $K_8^r(\mu)$ if we neglect the unknown contributions from $\mathcal{L}_4^{(Q)}$ proportional to M_π^2 (see Appendix B),

$$\begin{aligned}
\Delta M_\pi^2 \quad = \quad & 2e^2 \frac{\mathrm{C}}{\mathrm{F}_o^2} - e^2 \frac{1}{16\pi^2} M_\pi^2 \left(3 \ln \frac{M_\pi^2}{\mu^2} - 4 \right) \\
& - e^2 \frac{1}{8\pi^2} \frac{\mathrm{C}}{\mathrm{F}_o^4} \left[M_\pi^2 \left(3 \ln \frac{M_\pi^2}{\mu^2} + 1 \right) + M_K^2 \ln \frac{M_K^2}{\mu^2} \right] \\
& - 16 e^2 \frac{\mathrm{C}}{\mathrm{F}_o^4} \left[(M_\pi^2 + 2M_K^2) L_4^r + M_\pi^2 L_5^r \right] + 8 e^2 M_K^2 K_8^r + O(e^2 M_\pi^2).
\end{aligned} \tag{53}$$

The values of $L_4^r(\mu)$ are at the three scalepoints $\mu = (0.5\,\mathrm{GeV};\ M_\rho;\ 1\,\mathrm{GeV})$,

$$L_4^r(\mu) = (0.1;\ -0.3;\ -0.5) \pm 0.5 \times 10^{-3}\,. \tag{54}$$

We put ΔM_π^2 equal to the observed value and $K_8^r(\mu)$ becomes

$$K_8^r(\mu) = -(1.0;\ 4.0;\ 5.6) \pm 1.7 \times 10^{-3}\,. \tag{55}$$

K_8^r is of the same order as the coefficients L_i^r [2] and as it was expected by the so-called naive chiral power counting [14]. Note that $K_8^r(\mu)$ has not the usual scale-dependence implied by (37), because we have dropped scale-dependent terms in (53). Now we assume that the K_i^r's which contribute to R_π and R_K have the upper limit

$$|K_i| \lesssim \frac{1}{16\pi^2} = 6.3 \times 10^{-3}\,. \tag{56}$$

The contributions from $R_\pi M_\pi^2 + R_K M_K^2$ to the squared mass difference $\Delta M_K^2 - \Delta M_\pi^2$ is smaller than

$$|R_\pi M_\pi^2 + R_K M_K^2| \ \lesssim\ 2.6 \times 10^{-3} (\mathrm{GeV})^2\ , \tag{57}$$

which is a large number. Similarly the contribution to the mass ratio $(\Delta M_K^2/\Delta M_\pi^2)$ is

$$\frac{|R_\pi M_\pi^2 + R_K M_K^2|}{\Delta M_\pi^2} \lesssim 2.1 \, . \tag{58}$$

On the basis of (56), large corrections to Dashen's theorem can therefore not be excluded.

3.3 Independent Approach

We give a further estimate by looking at the QCD mass difference of the kaon. In this calculation we put $m_d - m_u \neq 0$, but neglect corrections of order $e^2(m_d - m_u)$. The observed squared mass difference of the kaon can be divided into two parts,

$$(M_{K^\pm}^2 - M_{K^\circ}^2)_{exp.} = (M_{K^\pm}^2 - M_{K^\circ}^2)_{QCD} + (M_{K^\pm}^2 - M_{K^\circ}^2)_{e.m.} \, . \tag{59}$$

The same is true for the pions, but the QCD term is of $O[(m_d - m_u)^2]$, which is very small, numerically $(M_{\pi^\pm} - M_{\pi^\circ})_{QCD} = 0.17\,\mathrm{MeV}$ [2]. We put therefore

$$(M_{\pi^\pm}^2 - M_{\pi^\circ}^2)_{exp.} = (M_{\pi^\pm}^2 - M_{\pi^\circ}^2)_{e.m.}. \tag{60}$$

The electromagnetic part of the squared mass difference using the notation of (41) becomes

$$\Delta M_K^2 - \Delta M_\pi^2 = (\Delta M_K^2 - \Delta M_\pi^2)_{exp.} - (\Delta M_K^2)_{QCD} + O(e^2 m_q^2). \tag{61}$$

The experimental part is $(\Delta M_K^2 - \Delta M_\pi^2)_{exp.} = -5.25 \times 10^{-3}(\mathrm{GeV})^2$ [12]. We refer to Leutwyler [7] for the calculation of $(\Delta M_K^2)_{QCD}$ without using Dashen's theorem,

$$(\Delta M_K^2)_{QCD} = -(M_K^2 - M_\pi^2)\frac{m_d - m_u}{m_s - \hat{m}} \left[1 + \Delta_M + O(m_q^2)\right] . \tag{62}$$

$R = (m_s - \hat{m})/(m_d - m_u)$ is the ratio determined from the mass splittings in the baryon octet and from $\rho - \omega$ mixing [11] with the value $R = 43.7 \pm 2.7$. One way to obtain the correction Δ_M is an estimate of $\eta - \eta'$ mixing. The phenomenological informations [2, 15] indicate that the mixing angle is somewhere between 20 and 25 degrees. The range $|\theta_{\eta\eta'}| = 22^\circ \pm 4^\circ$ corresponds to a value of $\Delta_M = 0.0 \pm 0.12$. Inserting these uncertainties into the equation above, the QCD squared mass difference of the kaon becomes

$$(\Delta M_K^2)_{QCD} = -(5.19 \pm 1.01) \times 10^{-3}(\mathrm{GeV})^2, \tag{63}$$

and the electromagnetic part of the squared mass difference is therefore (we neglect the errors in the experimental values)

$$\Delta M_K^2 - \Delta M_\pi^2 = -(0.06 \pm 1.01) \times 10^{-3}(\mathrm{GeV})^2 \tag{64}$$

with a large uncertainty for a small value. Similarly the mass ratio $(\Delta M_K^2/\Delta M_\pi^2)$ is

$$\frac{\Delta M_K^2}{\Delta M_\pi^2} = 0.95 \pm 0.81 \, . \tag{65}$$

3.4 Comparison with Other Results

Maltman and Kotchan [16] used the effective lagrangian $\mathcal{L}_2^{(Q)}$ and calculated the one-loop contributions to the squared mass differences $\Delta M^2_{\pi,K}$. They consider terms proportional to $e^2 M_\pi^2 \ln(M_\pi^2/\mu^2)$ and $e^2 M_K^2 \ln(M_K^2/\mu^2)$ at the scale point $\mu = 1.1\,\mathrm{GeV}$. The scale independent terms $e^2 M_\pi^2$ and $e^2 M_K^2$ as well as the contributions from the effective lagrangian of order p^4 are neglected. Their result is

$$(\Delta M_K^2 - \Delta M_\pi^2)_{log} = 0.76 \times 10^{-3} (\mathrm{GeV})^2, \tag{66}$$

which agrees with the contribution from the logarithms in our calculation at $\mu = 1.1\,\mathrm{GeV}$,

$$(\Delta M_K^2 - \Delta M_\pi^2)_{log} = 0.77 \times 10^{-3} (\mathrm{GeV})^2. \tag{67}$$

The authors expect that the ratio $(\Delta M_K^2/\Delta M_\pi^2)$ is in the range (at this scale point)

$$\frac{\Delta M_K^2}{\Delta M_\pi^2} = 1.44 \pm 0.20. \tag{68}$$

Using again $\Delta M_\pi^2 = (\Delta M_\pi^2)_{exp.}$, this corresponds to

$$\Delta M_K^2 - \Delta M_\pi^2 = (0.55 \pm 0.25) \times 10^{-3} (\mathrm{GeV})^2. \tag{69}$$

Recently two other estimates have been made by Donoghue, Holstein and Wyler [17] and Bijnens [18]. The first work is based on chiral symmetry and vector meson dominance. Their result shows a total ratio which is rather large,

$$\frac{\Delta M_K^2}{\Delta M_\pi^2} = 1.8\ . \tag{70}$$

The value for ΔM_π^2 they obtain at this order is about 20% higher than the experimental value,

$$M^2_{\pi^\pm} - M^2_{\pi^\circ} = 2 M_\pi \times 5.6\,\mathrm{MeV}. \tag{71}$$

Inserting (71) into the ratio above (70), the electromagnetic part of the squared mass difference becomes

$$\Delta M_K^2 - \Delta M_\pi^2 = 1.23 \times 10^{-3} (\mathrm{GeV})^2. \tag{72}$$

This result is in good correspondence with the one obtained by Bijnens [18],

$$\Delta M_K^2 - \Delta M_\pi^2 = (1.3 \pm 0.4) \times 10^{-3} (\mathrm{GeV})^2, \tag{73}$$

who calculated this value by splitting the contributions to the masses into two parts. A long distance part is estimated using the $(1/N_c)$-approach and a short distance contribution is determined in terms of the coupling constants L_i^r of the lagrangian $\mathcal{L}_4^{(Q)}$.

The results we listed in this section show rather large corrections to Dashen's theorem. They coincide only partly with the result we obtained in (64), but they could be in agreement with the estimate on the basis of (56). In order to establish them, we would have to numerically evaluate the relevant coefficients K_i^r of $\mathcal{L}_4^{(Q)}$.

4 Summary

1. We have evaluated the divergent part of the generating functional of CHPT including virtual photons to one loop. We have calculated the structure of the local action at order p^4 with 29 coupling constants $(L_1, \ldots, L_{10}; H_1, H_2; K_1, \ldots, K_{17})$, which have in general a single pole in $d = 4$ dimensions.

2. In a next step we have calculated the one-loop contribution to the meson masses in the isospin limit $m_u = m_d = \hat{m}$ and extracted the correction to Dashen's theorem [8],

$$\begin{aligned}
\Delta M_K^2 - \Delta M_\pi^2 \quad = \quad & -e^2 \frac{1}{16\pi^2}\left[3M_K^2 \ln\frac{M_K^2}{\mu^2} - 3M_\pi^2 \ln\frac{M_\pi^2}{\mu^2} - 4(M_K^2 - M_\pi^2)\right] \\
& -e^2 \frac{1}{8\pi^2}\frac{\mathrm{C}}{\mathrm{F}_\mathrm{o}^4}\left[M_K^2 \ln\frac{M_K^2}{\mu^2} - M_\pi^2(2\ln\frac{M_\pi^2}{\mu^2} + 1)\right] \\
& -16e^2 \frac{\mathrm{C}}{\mathrm{F}_\mathrm{o}^4} L_5^r(\mu)(M_K^2 - M_\pi^2) + R_\pi(\mu)M_\pi^2 + R_K(\mu)M_K^2 \\
& +O(e^2 m_q^2),
\end{aligned} \tag{74}$$

where $\Delta M_P^2 = M_{P^\pm}^2 - M_{P^\mathrm{o}}^2$ with P the pseudoscalar meson in question. $L_5^r(\mu)$ and $R_{\pi,K}$ are the contributions from $\mathcal{L}_4^{(Q)}$,

$$\begin{aligned}
R_\pi &= \frac{2}{3}e^2\left(6K_3^r - 3K_4^r + 2K_9^r - 10K_{10}^r - 12K_{11}^r\right), \\
R_K &= -\frac{4}{3}e^2\left(K_5^r + K_6^r - 6K_{10}^r - 6K_{11}^r\right).
\end{aligned} \tag{75}$$

At the three scale points $\mu = (0.5;\ 0.77;\ 1)$ GeV we got numerically

$$\Delta M_K^2 - \Delta M_\pi^2 = (-0.26;\, 0.49;\, 0.99) \pm 0.13 \times 10^{-3}(\mathrm{GeV})^2 \ + R_\pi M_\pi^2 + R_K M_K^2. \tag{76}$$

We assumed the upper limit of $R_\pi M_\pi^2 + R_K M_K^2$ to be

$$|R_\pi M_\pi^2 + R_K M_K^2| \ \lesssim\ 2.6 \times 10^{-3}(\mathrm{GeV})^2 , \tag{77}$$

from which we can not exclude large corrections to Dashen's theorem. Note that in the numerical part in (76) the correction due to $L_5^r(\mu)$ is already included.

3. In addition, we have given an estimate of $\Delta M_K^2 - \Delta M_\pi^2$ on the basis of ref.[7],

$$\Delta M_K^2 - \Delta M_\pi^2 = -(0.06 \pm 1.01) \times 10^{-3}(\mathrm{GeV})^2, \tag{78}$$

a small number with a large uncertainty.

4. We have compared our values to the results in the literature [16, 17, 18]. We found that

they coincide only partly with the result (78), but they could be well in agreement with the estimate in (76). In order to complete the expression for $\Delta M_K^2 - \Delta M_\pi^2$ we need the numerical evaluation of $R_{\pi,K}$, i.e the calculation of the finite parts of the coupling constants K_i. This is beyond the scope of this work.

Acknowledgements

I am grateful to Jürg Gasser for many discussions and for reading this manuscript carefully. Also I thank H. Neufeld who pointed out, that my original lagrangian $\mathcal{L}_4^{(Q)}$ contains an overcomplete set of counterterms, J. Bijnens for correspondence concerning effective lagrangians, Urs Bürgi, who helped to draw the Feynman graphs, and Robert Baur for pointing out a numerical error in the coefficients Σ_i, that he had worked out independently.

A Matrix Relations

For the derivation of the effective lagrangian $\mathcal{L}_4^{(Q)}$ at order p^4, we used the equation of motion obeyed by $\bar{U}$,

$$\begin{aligned} & d^\mu d_\mu U^+ U - U^+ d^\mu d_\mu U + (U^+\chi - \chi^+ U) \\ & \quad +4\frac{\mathrm{C}}{\mathrm{F}_\mathrm{o}^2}(U^+QUQ - QU^+QU) - \frac{1}{3} < U^+\chi - \chi^+ U >= 0. \end{aligned} \tag{79}$$

The Cayley - Hamilton theorem for a 3×3-matrix A,

$$\begin{aligned} & A^3 - < A > A^2 + \frac{1}{2}\left\{< A >^2 - < A^2 >\right\} A \\ & \quad -\frac{1}{6}\left\{< A >^3 - 3 < A >< A^2 > + 2 < A^3 >\right\} \mathbf{1} = 0, \end{aligned} \tag{80}$$

implies a trace identity for 3×3-matrices A, B, C and D,

$$\begin{aligned} < ABCD > \;=\; & - < ABDC > - < ACBD > - < ACDB > \\ & - < ADBC > - < ADCB > + < AB >< CD > \\ & + < AC >< BD > + < AD >< BC > + < A >< BCD > \\ & + < A >< BDC > + < B >< ACD > + < B >< ADC > \\ & + < C >< ABD > + < C >< ADB > + < D >< ABC > \\ & + < D >< ACB > - < A >< B >< CD > \\ & - < A >< C >< BD > - < A >< D >< BC > \\ & - < B >< C >< AD > - < B >< D >< AC > \\ & - < C >< D >< AB > + < A >< B >< C >< D > . \end{aligned} \tag{81}$$

Furthermore we used the observation that Q^2 can be written as

$$Q^2 = \frac{e}{3}Q + \frac{2e^2}{9}\mathbf{1}. \tag{82}$$

We confirmed the 17 independent terms $K_i < \cdots >$ $(i = 1, 2, \ldots, 17)$ in $\mathcal{L}_4^{(Q)}$ calculating the contributions from tree graphs at order p^4 to

- the constant term $e^4\mathbf{1}$,
- the masses $M^2_{\pi^\pm}, M^2_{\pi^{\rm o}}$ and M^2_η,
- the scattering amplitudes at order $e^2 m_q$

$$
\begin{aligned}
\pi^+\pi^- &\rightarrow \pi^+\pi^-, \\
\pi^+\pi^- &\rightarrow K^{\rm o}\bar{K}^{\rm o}, \\
K^+K^- &\rightarrow K^{\rm o}\bar{K}^{\rm o},
\end{aligned}
\tag{83}
$$

- the scattering amplitudes at order e^4

$$
\begin{aligned}
\pi^+\pi^- &\rightarrow \pi^+\pi^-, \\
\pi^+\pi^- &\rightarrow K^{\rm o}\bar{K}^{\rm o},
\end{aligned}
\tag{84}
$$

- the matrix elements

$$
\begin{aligned}
&< \pi^+|\bar{u}\gamma_5 d|\pi^+\pi^- >, \quad < \pi^+|\bar{u}\gamma_5 d|\eta\eta >, \\
&< 0|\bar{u}\gamma_\mu\gamma_5 d|\pi^- >, \quad < 0|T\,\bar{u}(x)\gamma^\mu\gamma_5 d(x)\bar{d}(y)\gamma_\mu\gamma_5 u(y)|0 >, \\
&< \pi^+|T\,\bar{u}(x)\gamma^\mu d(x)\bar{d}(y)\gamma_\mu u(y)|\pi^+ >, \quad < 0|T\,\bar{u}(x)\gamma^\mu d(x)\bar{d}(y)\gamma_\mu u(y)|0
\end{aligned}
\tag{85}
$$

B The Formal Expressions for the Masses at Order $e^2 m_q$

The photon loop (see fig.1) gives a contribution to the masses of the charged particles,

$$
\delta M^2_{C^\pm}|_{\gamma-loop} = -e^2 \frac{1}{16\pi^2} M_C^2 \left(3 \ln \frac{M_C^2}{\mu^2} - 4 \right), \tag{86}
$$

where C stands for π and K, respectively.

From the tadpole, the masses of the charged particles get a change of the form

$$
\delta M^2_{C^\pm}|_{tadpole} = -e^2 \frac{1}{8\pi^2} \frac{\rm C}{\rm F_o^4} \left[A_{C^\pm} M_\pi^2 \ln \frac{M_\pi^2}{\mu^2} + B_{C^\pm} M_K^2 \ln \frac{M_K^2}{\mu^2} \right], \tag{87}
$$

with $(A_{\pi^\pm},\, B_{\pi^\pm}) = (2,\, 1)$ and $(A_{K^\pm},\, B_{K^\pm}) = (1,\, 2)$. The contributions to the masses of $\pi^{\rm o}$ and η are

$$
\delta M^2_P|_{tadpole} = e^2 \frac{1}{48\pi^2} \frac{\rm C}{\rm F_o^4} \left[\alpha_P M_\pi^2 \left(\ln \frac{M_\pi^2}{\mu^2} + 1 \right) + \beta_P M_K^2 \left(\gamma_P \ln \frac{M_K^2}{\mu^2} + 1 \right) \right], \tag{88}
$$

with the coefficients $(\alpha_{\pi^{\rm o}},\, \beta_{\pi^{\rm o}},\, \gamma_{\pi^{\rm o}}) = (6,\, 0,\, 0)$ and $(\alpha_\eta,\, \beta_\eta,\, \gamma_\eta) = (-2,\, 1,\, 4)$. The masses of the neutral kaons do not get contributions from the loops. At order $e^2 m_q$, the mass-shift due to $\mathcal{L}_4^{(Q)}$

involves in general only terms with the couplings K_i, but for the charged particles, where L_4 and L_5 enter as well,

$$\delta M^2_{C^\pm}|_{\mathcal{L}_4^{(Q)}} = -16e^2 \frac{\mathrm{C}}{\mathrm{F}_\mathrm{o}^4} \left[(M_\pi^2 + 2M_K^2) L_4^r(\mu) + M_C^2 L_5^r(\mu)\right] + \cdots . \tag{89}$$

The term proportional to $L_4^r(\mu)$ drops out in the difference $\Delta M_K^2 - \Delta M_\pi^2$. Explicitly, the masses are at $O(e^2 m_q)$,

$$
\begin{aligned}
M^2_{\pi^\pm} &= \hat{M}^2_{\pi^\pm} - e^2 \frac{1}{16\pi^2} M^2_\pi \left(3 \ln \frac{M^2_\pi}{\mu^2} - 4 \right) \\
&\quad - e^2 \frac{1}{8\pi^2} \frac{\mathrm{C}}{\mathrm{F}^4_\mathrm{o}} \left[2M^2_\pi \ln \frac{M^2_\pi}{\mu^2} + M^2_K \ln \frac{M^2_K}{\mu^2} \right] \\
&\quad - 16 e^2 \frac{\mathrm{C}}{\mathrm{F}^4_\mathrm{o}} \left[(M^2_\pi + 2M^2_K) L^r_4 + M^2_\pi L^r_5 \right] \\
&\quad - \frac{4}{9} e^2 M^2_\pi \, (6K^r_1 + 6K^r_2 + 5K^r_5 + 5K^r_6 - 6K^r_7 \\
&\qquad\qquad - 15K^r_8 - 5K^r_9 - 23K^r_{10} - 18K^r_{11}) \\
&\quad + 8e^2 M^2_K K^r_8 \, ,
\end{aligned}
\tag{90}
$$

$$
\begin{aligned}
M^2_{\pi^\circ} &= \hat{M}^2_{\pi^\circ} + e^2 \frac{1}{8\pi^2} \frac{\mathrm{C}}{\mathrm{F}^4_\mathrm{o}} M^2_\pi \left(\ln \frac{M^2_\pi}{\mu^2} + 1 \right) \\
&\quad - \frac{2}{9} e^2 M^2_\pi \, (12K^r_1 + 12K^r_2 - 18K^r_3 + 9K^r_4 + 10K^r_5 \\
&\qquad\qquad + 10K^r_6 - 12K^r_7 - 12K^r_8 - 10K^r_9 - 10K^r_{10}) \, ,
\end{aligned}
\tag{91}
$$

$$
\begin{aligned}
M^2_{K^\pm} &= \hat{M}^2_{K^\pm} - e^2 \frac{1}{16\pi^2} M^2_K \left(3 \ln \frac{M^2_K}{\mu^2} - 4 \right) \\
&\quad - e^2 \frac{1}{8\pi^2} \frac{\mathrm{C}}{\mathrm{F}^4_\mathrm{o}} \left[M^2_\pi \ln \frac{M^2_\pi}{\mu^2} + 2M^2_K \ln \frac{M^2_K}{\mu^2} \right] \\
&\quad - 16 e^2 \frac{\mathrm{C}}{\mathrm{F}^4_\mathrm{o}} \left[(M^2_\pi + 2M^2_K) L^r_4 + M^2_K L^r_5 \right] \\
&\quad + \frac{4}{3} e^2 M^2_\pi \, (3K^r_8 + K^r_9 + K^r_{10}) \\
&\quad - \frac{4}{9} e^2 M^2_K \, (6K^r_1 + 6K^r_2 + 5K^r_5 + 5K^r_6 - 6K^r_7 \\
&\qquad\qquad - 24K^r_8 - 2K^r_9 - 20K^r_{10} - 18K^r_{11}) \, ,
\end{aligned}
\tag{92}
$$

$$
\begin{aligned}
M^2_{K^\circ} &= M^2_{\bar{K}^\circ} \\
&= \hat{M}^2_{K^\circ} - \frac{8}{9} e^2 M^2_K \, (3K^r_1 + 3K^r_2 + K^r_5 + K^r_6 \\
&\qquad\qquad - 3K^r_7 - 3K^r_8 - K^r_9 - K^r_{10}) \, ,
\end{aligned}
\tag{93}
$$

$$
M^2_\eta = \hat{M}^2_\eta - e^2 \frac{1}{48\pi^2} \frac{\mathrm{C}}{\mathrm{F}^4_\mathrm{o}} \left[2M^2_\pi \left(\ln \frac{M^2_\pi}{\mu^2} + 1 \right) - M^2_K \left(4 \ln \frac{M^2_K}{\mu^2} + 1 \right) \right]
$$

$$
\begin{aligned}
&+\frac{4}{9}e^2 M_\pi^2\,(K_9^r + K_{10}^r) \\
&-\frac{2}{9}e^2 M_\eta^2\,(12K_1^r + 12K_2^r - 6K_3^r + 3K_4^r + 6K_5^r \\
&\qquad\qquad +6K_6^r - 12K_7^r - 12K_8^r - 4K_9^r - 4K_{10}^r)\,. \qquad (94)
\end{aligned}
$$

The squared mass differences of the pions and the kaons are

$$
\begin{aligned}
\Delta M_\pi^2 &= M_{\pi^\pm}^2 - M_{\pi^\mathrm{o}}^2 \\
&= 2e^2\frac{\mathrm{C}}{\mathrm{F_o^2}} - e^2\frac{1}{16\pi^2}M_\pi^2\left(3\ln\frac{M_\pi^2}{\mu^2} - 4\right) \\
&\qquad -e^2\frac{1}{8\pi^2}\frac{\mathrm{C}}{\mathrm{F_o^4}}\left[M_\pi^2\left(3\ln\frac{M_\pi^2}{\mu^2} + 1\right) + M_K^2\ln\frac{M_K^2}{\mu^2}\right] \\
&\qquad -16e^2\frac{\mathrm{C}}{\mathrm{F_o^4}}\left[(M_\pi^2 + 2M_K^2)L_4^r + M_\pi^2 L_5^r\right] \\
&\qquad +2e^2 M_\pi^2\,(-2K_3^r + K_4^r + 2K_8^r + 4K_{10}^r + 4K_{11}^r) \\
&\qquad +8e^2 M_K^2 K_8^r\;, \qquad (95)
\end{aligned}
$$

$$
\begin{aligned}
\Delta M_K^2 &= M_{K^\pm}^2 - M_{K^\mathrm{o}}^2 \\
&= 2e^2\frac{\mathrm{C}}{\mathrm{F_o^2}} - e^2\frac{1}{16\pi^2}M_K^2\left(3\ln\frac{M_K^2}{\mu^2} - 4\right) \\
&\qquad -e^2\frac{1}{8\pi^2}\frac{\mathrm{C}}{\mathrm{F_o^4}}\left[M_\pi^2\ln\frac{M_\pi^2}{\mu^2} + 2M_K^2\ln\frac{M_K^2}{\mu^2}\right] \\
&\qquad -16e^2\frac{\mathrm{C}}{\mathrm{F_o^4}}\left[(M_\pi^2 + 2M_K^2)L_4^r + M_K^2 L_5^r\right] \\
&\qquad +\frac{4}{3}e^2 M_\pi^2\,(3K_8^r + K_9^r + K_{10}^r) \\
&\qquad -\frac{4}{3}e^2 M_K^2\,(K_5^r + K_6^r - 6K_8^r - 6K_{10}^r - 6K_{11}^r)\,. \qquad (96)
\end{aligned}
$$

And finally the difference $\Delta M_K^2 - \Delta M_\pi^2$ is

$$
\begin{aligned}
\Delta M_K^2 - \Delta M_\pi^2 &= -e^2\frac{1}{16\pi^2}\left[3M_K^2\ln\frac{M_K^2}{\mu^2} - 3M_\pi^2\ln\frac{M_\pi^2}{\mu^2} - 4(M_K^2 - M_\pi^2)\right] \\
&\qquad -e^2\frac{1}{8\pi^2}\frac{\mathrm{C}}{\mathrm{F_o^4}}\left[M_K^2\ln\frac{M_K^2}{\mu^2} - M_\pi^2(2\ln\frac{M_\pi^2}{\mu^2} + 1)\right] \\
&\qquad -16e^2\frac{\mathrm{C}}{\mathrm{F_o^4}}L_5^r(\mu)(M_K^2 - M_\pi^2) + R_\pi(\mu)M_\pi^2 + R_K(\mu)M_K^2\,, \qquad (97)
\end{aligned}
$$

where $R_{\pi,K}$ are,

$$
\begin{aligned}
R_\pi &= \frac{2}{3}e^2\,(6K_3^r - 3K_4^r + 2K_9^r - 10K_{10}^r - 12K_{11}^r)\,, \\
R_K &= -\frac{4}{3}e^2\,(K_5^r + K_6^r - 6K_{10}^r - 6K_{11}^r)\,. \qquad (98)
\end{aligned}
$$

References

[1] S.Weinberg, Physica **96A** (1979) 327

[2] J.Gasser, H.Leutwyler, Nucl. Phys. **B250** (1985) 465

[3] J.Kambor, J.Missimer, D.Wyler, Nucl. Phys. **B346** (1990) 17

[4] H.Leutwyler, Rapporteur talk given at the XXVI Int. Conf. on High Energy Physics, Dallas, 1992, edited by J.R. Sanford, AIP Conf. Proc. No. 272 (AIP, New York, 1993); *On the Foundations of Chiral Perturbation Theory*, preprint BUTP-93/24, to appear in Ann. of Phys. (N.Y.);
U.-G. Meißner, Rep. Prog. Phys. **56** (1993) 903;
A.Pich, *Introduction to Chiral Perturbation Theory*, Lectures given at the V Mexican School of Particles and Fields, Guanajuato, México, December 1992, preprint CERN-TH. 6978/93;
G.Ecker, *The Standard Model at Low Energies*, Lectures given at the 6$^{\mathrm{th}}$ Indian-Summer School on Intermediate Energy Physics Interaction in Hadronic Systems, Prague, August 25 - 31, 1993, preprint UWThPh-1993-31, to appear in the Proceedings (Czech. J. Phys)

[5] G.Ecker, J.Gasser, A.Pich, E.de Rafael, Nucl. Phys. **B321** (1989) 311

[6] J.Gasser, H.Leutwyler, Ann. of Phys. (N.Y.) **158** (1984) 142

[7] H.Leutwyler, Nucl. Phys. **B337** (1990) 108

[8] R.Dashen, Phys. Rev. **183** (1969) 1245

[9] T.Das, G.S.Guralnik, V.S.Mathur, F.E.Low, J.E.Young, Phys. Rev. Lett. **18** (1967) 759

[10] S.Weinberg, Phys. Rev. Lett. **18** (1967) 507

[11] J.Gasser, H.Leutwyler, Phys. Rep. **87** (1982) 77

[12] Particle Data Group, K.Hikasa et al., Phys. Rev. **D45** (1992) S1

[13] R.D.Peccei, J.Solà, Nucl. Phys. **B281** (1987) 1

[14] A.Manohar, H.Georgi, Nucl. Phys. **B234** (1984) 189

[15] J.F.Donoghue, B.R.Holstein, Y.-C.R.Lin, Phys. Rev. Lett. **55** (1985) 2766;
F.J.Gilman, R.Kauffman, Phys. Rev. **D36** (1987) 2761;
Riazuddin, Fayyazuddin, Phys. Rev. **D37** (1988) 149

[16] K.Maltman, D.Kotchan, Mod. Phys. Lett. **A5** (1990) 2457

[17] J.F.Donoghue, B.R.Holstein, D.Wyler, Phys. Rev. **D47** (1993) 2089

[18] J.Bijnens, Phys. Lett. **B306** (1993) 343